SUPERCONGRUENCES FOR THE ALMKVIST-ZUDILIN NUMBERS

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Abstract. Given a prime number p, the study of divisibility properties of a sequence $c(n)$ has two contending approaches: p-adic valuations and superconcongruences. The former searches for the highest power of p dividing $c(n)$, for each n; while the latter (essentially) focuses on the maximal powers r and t such that $c(p^r n)$ is congruent to $c(p^{r-1}n)$ modulo p^t. This is called supercongruence. In this paper, we prove a conjecture on supercongruences for sequences that have come to be known as the Almkvist-Zudilin numbers. Some other (naturally) related family of sequences will be considered in a similar vain.

1. Introduction

The *Apéry numbers* $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ were valuable to R. Apéry in his celebrated proof [1] that $\zeta(3)$ is an irrational number. Since then these numbers have been a subject of much research. For example, they stand among a host of other sequences with the property

$$A(p^r n) \equiv_{p^{3r}} A(p^{r-1}n)$$

now known as *supercongruence* − a term dubbed by F. Beukers [2].

At the heart of many of these congruences sits the classical example $\binom{pb}{pc} \equiv_{p^3} \binom{b}{c}$ which is a stronger variant of the famous congruence $\binom{pb}{pc} \equiv_p \binom{b}{c}$ of Lucas. For a compendium of references on the subject of Apéry-type sequences, see [9].

Let us begin by fixing notational conventions. Denote the set of positive integers by $\mathbb{N}^+$. For $m \in \mathbb{N}^+$, let $\equiv_m$ represent congruence modulo m. Throughout, assume $p \geq 5$ is a prime.

In this paper, true to tradition, we aim to investigate similar type of supercongruences for the following family of sequences. For integers $i \geq 0$ and $n \geq 1$, define

$$a_i(n) := \sum_{k=0}^{\lfloor (n-i)/3 \rfloor} (-1)^{n-k} \binom{3k+i}{k} \binom{2k+i}{k} \binom{n}{3k+i} \binom{n+k}{k} 3^{n-3k-i}$$

In recent literature, $a_0(n)$ are referred to as the Almkvist-Zudilin numbers. Our motivation for the present work here emanates from the following claim found in [6] (see also [3], [7]).

Conjecture 1.1. *For a prime p and $n \in \mathbb{N}^+$, the Almkvist-Zudilin numbers satisfy*

$$a_0(pn) \equiv_{p^3} a_0(n).$$

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Our main results can be summed up as:

if p is a prime and $n \in \mathbb{N}^+$, then $a_0(pn) \equiv_{p^3} a_0(n)$ and $a_i(pn) \equiv_{p^2} 0$ for $i > 0$.

The organization of the paper is as follows. Section 2 lays down some preparatory results to show the vanishing of $a_i(pn)$ modulo p^2, for $i > 0$. Section 3 sees the completion of the proof. Our principal approach in proving the main conjecture $a_0(pn) \equiv_{p^3} a_0(n)$ relies on a "machinery" we develop as a proof strategy which maybe described schematically as:

$$\textit{reduction} + p\textit{-identities}.$$

Sections 4 and 5 exhibit its elaborate execution. The reduction brings in a *tighter* claim and it also offers an advantage in allowing to work with a single sum instead of a double sum. In Section 6, we complete the proof for Conjecture 1.1. The paper concludes with Section 7 where we declare an improvement on the results from Section 3 which states a congruence for the family of sequences $a_i(pn)$ modulo p^3, when $i > 0$. Furthermore, in this last section, the reader will find a proof outline guided by our "machinery".

2. Preliminary results

Fermat quotients are numbers of the form $q_p(x) = \frac{x^{p-1}-1}{p}$ and they played a useful role in the study of cyclotomic fields and Fermat's Last Theorem, see [8]. The next three lemmas are known and we give their proofs for the sake of completeness.

Lemma 2.1. *If $a \not\equiv_p 0$ then for $d \in \mathbb{Z}$,*

$$q_p(a^d) \equiv_{p^2} d\, q_p(a) + p\binom{d}{2} q_p(a)^2. \tag{2.1}$$

Proof. Since by Fermat's little theorem $a^{p-1} \equiv_p 1$ then it follows that

$$\left(a^{p-1}\right)^d = \left(1 + (a^{p-1} - 1)\right)^d \equiv_{p^3} 1 + d(a^{p-1} - 1) + \binom{d}{2}(a^{p-1} - 1)^2.$$

□

Lemma 2.2. *Let $H_n = \sum_{j=1}^n \frac{1}{j}$ be the n-th harmonic number. Then, for $n \in \mathbb{N}^+$, we have*

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\frac{1}{k} = -2H_n. \tag{2.2}$$

Proof. For an indeterminate y, a simple partial fraction decomposition proves the identity (see [5, Lemma 3.1])

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\frac{1}{k+y} = \frac{(-1)^n}{y}\prod_{j=1}^{n}\frac{y-j}{y+j}. \tag{2.3}$$

Now, subtract $\frac{1}{y}$ from both sides and take the limit as $y \to 0$. The right-hand side takes the form

$$\frac{1}{n!}\lim_{y\to 0}\left[\frac{\prod_{j=1}^n (j-y) - \prod_{j=1}^n (j+y)}{y}\right] = -2\sum_{k=1}^{n}\frac{1}{k}.$$

The conclusion is clear. □

Lemma 2.3. *Suppose p is a prime and $0 \leq k < p/3$. Then,*

$$(-1)^k \binom{\lfloor p/3 \rfloor}{k} \binom{\lfloor p/3 \rfloor + k}{k} \equiv_p \binom{3k}{k,k,k} 3^{-3k}.$$

Proof. We observe that $\binom{n}{k}\binom{n+k}{k} = \binom{2k}{k}\binom{n+k}{2k}$. If $p \equiv_3 1$, then $\lfloor \frac{p}{3} \rfloor = \frac{p-1}{3}$ and hence

$$\begin{aligned}\binom{\frac{p-1}{3}+k}{2k} &= \frac{\frac{p-1}{3}(\frac{p-1}{3}+k)}{(2k)!}\prod_{j=1}^{k-1}\left(\frac{p-1}{3} \pm j\right) \\ &\equiv_p \frac{(-1)^k(3k-1)}{3^{2k}(2k)!}\prod_{j=1}^{k-1}(3j \pm 1) = \frac{(-1)^k(3k)!}{3^{3k}(2k)!k!}.\end{aligned}$$

Therefore, we gather that

$$(-1)^k \binom{\frac{p-1}{3}}{k}\binom{\frac{p-1}{3}+k}{k} = (-1)^k \binom{2k}{k}\binom{\frac{p-1}{3}+k}{2k} \equiv_p \frac{(3k)!}{3^{3k}!k!^3} = \binom{3k}{k,k,k}3^{-3k}.$$

The case $p \equiv_3 -1$ runs analogously. □

Corollary 2.4. *For a prime p and an integer $0 < i < \frac{p}{3}$, we have the congruences*

$$\sum_{k=1}^{p-1} \binom{3k}{k,k,k}\frac{3^{-3k}}{k} \equiv_p \sum_{k=1}^{\lfloor p/3 \rfloor} \binom{3k}{k,k,k}\frac{3^{-3k}}{k} \equiv_p 3q_p(3),$$

$$\sum_{k=0}^{p-1} \binom{3k}{k,k,k}\frac{3^{-3k}}{k+i} \equiv_p \sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k}{k,k,k}\frac{3^{-3k}}{k+i} \equiv_p 0.$$

Proof. For the first assertion, we combine (2.2), Lemma 2.3 and the congruence ([4, p. 358])

$$H_{\lfloor p/3 \rfloor} \equiv_p -3\sum_{r=1}^{\lfloor p/3 \rfloor}\frac{1}{p-3r} \equiv_p -\frac{3q_p(3)}{2}.$$

The second congruence follows from (2.3) with $y = i$ and Lemma 2.3. □

3. Main results on the sequences $a_i(n)$ for $i > 0$

Theorem 3.1. *For a prime p and $n, i \in \mathbb{N}^+$ with $i < \frac{p}{3}$, we have $a_i(pn) \equiv_{p^2} 0$.*

Proof. Let $k = pm + r$ for $0 \leq r \leq p-1$. Note: $3k + i = 3pm + 3r + i \leq pn$. Write

$$\begin{aligned}a_i(pn) = \sum_{m=0}^{\lfloor n/3 \rfloor}\sum_{r=0}^{p-1}(-1)^{pn-pm-r}&\binom{3pm+3r+i}{pm+r}\binom{2pm+2r+i}{pm+r} \\ &\cdot \binom{pn}{3pm+3r+i}\binom{pn+pm+r}{pm+r}3^{pn-3pm-3r-i}.\end{aligned}$$

If $t := 3r + i \geq p + 1$, it is easy to show that the following terms vanish modulo p^2:

$$\binom{3pm+t}{pm+r}\binom{2pm+2r+i}{pm+r}\binom{pn}{3pm+t} = \binom{3pm+t}{pm+r,pm+r,pm+r+i}\binom{pn}{3pm+t}.$$

Therefore, we may restrict to the remaining sum with $3r+i\le p$:

$$a_i(pn)=\sum_{m=0}^{\lfloor n/3\rfloor}\sum_{r=0}^{\lfloor (p-i)/3\rfloor}(-1)^{n-m-r}\binom{3pm+3r+i}{pm+r}\binom{2pm+2r+i}{pm+r}\cdot\binom{pn}{3pm+3r+i}\binom{pn+pm+r}{pm+r}3^{pn-3pm-3r-i}.$$

We need Lucas's congruence $\binom{pb+c}{pd+e}\equiv_p\binom{d}{d}\binom{c}{e}$ to arrive at

$$a_i(pn)\equiv_p\sum_{m=0}^{\lfloor n/3\rfloor}\sum_{r=0}^{\lfloor (p-i)/3\rfloor}(-1)^{n-m-r}\binom{3m}{m}\binom{3r+i}{r}\binom{2m}{m}\binom{2r+i}{r}\cdot\binom{pn}{3pm+3r+i}\binom{n+m}{m}3^{pn-3pm-3r-1}.$$

For $0<j<p$, we apply Gessel's congruence $\binom{p}{j}\equiv_{p^2}(-1)^{j-1}\frac{p}{j}$ (if $p=3r+i$, in this case, still the corresponding term properly absorbs into the sum below) so that

$$\binom{pn}{3pm+3r+i}=\frac{pn}{3pm+3r+i}\binom{pn-1}{3pm+3r+i-1}=\frac{pn}{3pm+3r+i}\binom{p(n-1)+p-1}{3pm+3r+i-1}$$
$$\equiv_{p^2}(-1)^{r+i-1}\frac{pn}{3r+i}\binom{n-1}{3m},$$

which leads to

$$a_i(pn)\equiv_{p^2}pn\sum_{m=0}^{\lfloor n/3\rfloor}\sum_{r=0}^{\lfloor (p-i)/3\rfloor}(-1)^{n-m-r}\binom{3m}{m}\binom{3r+i}{r}\binom{2m}{m}\binom{2r+i}{r}\cdot\frac{(-1)^{r+i-1}}{3r+i}\binom{n-1}{3m}\binom{n+m}{m}3^{pn-3pm-3r-i}.$$

Next, we use Fermat's Little Theorem and *decouple* the double sum to obtain

$$a_i(pn)\equiv_{p^2}n\sum_{m=0}^{\lfloor n/3\rfloor}(-1)^{n-m+i-1}3^{n-3m-i}\binom{3m}{m}\binom{2m}{m}\binom{n-1}{3m}\binom{n+m}{m}\cdot p\sum_{r=0}^{\lfloor (p-i)/3\rfloor}\binom{3r+i}{r}\binom{2r+i}{r}\frac{3^{-3r}}{3r+i}.$$

It suffices to verify the *sum over* r vanishes modulo p. To achieve this, apply partial fraction decomposition and Corollary 2.4 (upgrading the sum to $\lfloor p/3\rfloor$ is *harmless* here). Thus,

$$\sum_{k=0}^{\lfloor p/3\rfloor}\binom{3k+i}{k}\binom{2k+i}{i}\frac{3^{-3k}}{3k+i}=\sum_{k=0}^{\lfloor p/3\rfloor}\binom{3k}{k,k,k}3^{-3k}\prod_{j=1}^{i-1}(3k+j)\prod_{j=1}^{i}(k+j)^{-1}$$
$$=\sum_{j=1}^{i}\alpha_j(i)\sum_{k=0}^{\lfloor p/3\rfloor}\binom{3k}{k,k,k}\frac{3^{-3k}}{k+j}\equiv_p\sum_{j=1}^{i}\alpha_j(i)\cdot 0=0;$$

where $\alpha_j(i)\in\mathbb{Q}$ are some constants. We have enough reason to conclude the proof. $\square$

4. The reduction on the sequence $a_0(n)$

Our proof of Conjecture 1.1 requires a slightly more delicate analysis than what has been demonstrated in the previous sections for the sequences $a_i(n)$, where $i > 0$. As a first major step forward, we state and prove the following *somewhat* stronger result. This will be crucial in scaling down a double sum, which emerges (see proof below) as an expression for the sequence $a_0(pn)$, to a single sum.

Theorem 4.1. *The congruence*

$$\sum_{r=1}^{p-1}(-1)^r\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r}\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}3^{-3r} \equiv_{p^3} p\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}q_p(3^{-(n-3m)}) \tag{4.1}$$

or

$$\sum_{r=0}^{p-1}(-1)^r\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r}\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}3^{-3r} \equiv_{p^3} \binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}3^{-(n-3m)(p-1)}$$

implies $a_0(pn) \equiv_{p^3} a_0(n)$.

Proof. Let $k = pm + r$ for $0 \le r < p$. Then, by using the new parameters,

$$a_0(pn) = \sum_{m=0}^{n-1}3^{p(n-3m)}(-1)^{n-m}\sum_{r=0}^{p-1}(-1)^r\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r} \cdot\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}3^{-3r}$$

Let's isolate the case $r = 0$, then, from $\binom{pb}{pc} \equiv_{p^3} \binom{b}{c}$ and the hypothesis we get

$$\begin{aligned} a_0(pn) &\equiv_{p^3} \sum_{m=0}^{n-1}3^{p(n-3m)}(-1)^{n-m}\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}\left[1+pq_p(3^{-(n-3m)})\right] \\ &\equiv_{p^3} \sum_{m=0}^{n-1}(-1)^{n-m}\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}3^{(n-3m)} = a_0(n). \end{aligned}$$

□

5. Further Preliminary results

In this section, we build a few valuable results aiming at the proof of Theorem 4.1 and hence that of Conjecture 1.1.

Lemma 5.1. *If $a > b \ge 0$ and $0 < j < p$ then*

$$\binom{ap}{bp+j} \equiv_{p^2} (a-b)\binom{a}{b}\binom{p}{j} \quad \text{and} \quad \binom{ap}{bp-j} \equiv_{p^2} b\binom{a}{b}\binom{p}{j}. \tag{5.1}$$

Moreover, for $0 \le r < p$,

$$\binom{p(n+m)+r}{pm+r} \equiv_{p^2} \binom{n+m}{m}\left(1+n\left(\binom{p+r}{r}-1\right)\right) \tag{5.2}$$

$$\binom{2pm+2r}{pm+r} \equiv_{p^2} \binom{2m}{m}\left(\binom{2r}{r}+2m\binom{p+2r}{r}-2m\binom{2r}{r}\right), \tag{5.3}$$

$$\begin{aligned}\binom{3pm+3r}{pm+r} \equiv_{p^2} &\binom{3m}{m}\left(2m\binom{p+3r}{r}+m\binom{p+3r}{2r}-(3m-1)\binom{3r}{r}\right)\\ &+\binom{3m}{m-1}\left(\binom{3r}{p+r}+(m-1)\binom{p+3r}{2p+r}-3m\binom{3r}{p+r}\right).\end{aligned} \tag{5.4}$$

Also, $\binom{pn}{3pm+3r} \equiv_{p^3} \frac{pn}{3pm+3r} U_r$ *where*

$$\begin{aligned}U_r \equiv_{p^2} &(3m+1)\binom{n-1}{3m+1}\left[\binom{2p-1}{3r-1}-\binom{p-1}{3r-1}-\binom{p-1}{3r-1-p}\right]\\ &+(3m+2)\binom{n-1}{3m+2}\left[\binom{2p-1}{3r-1-p}-\binom{p-1}{3r-1-p}-\binom{p-1}{3r-1-2p}\right]\\ &+(3m+3)\binom{n-1}{3m+3}\left[\binom{2p-1}{3r-1-2p}-\binom{p-1}{3r-1-2p}\right]\\ &+3m\binom{n-1}{3m}\left[\binom{2p-1}{p+3r-1}-\binom{p-1}{3r-1}\right]\\ &+\binom{n-1}{3m}\binom{p-1}{3r-1}+\binom{n-1}{3m+1}\binom{p-1}{3r-1-p}+\binom{n-1}{3m+2}\binom{p-1}{3r-1-2p}.\end{aligned} \tag{5.5}$$

Proof. For (5.1), we have

$$\binom{ap}{bp+j} = \binom{ap}{bp}\frac{(a-b)p}{bp+j}\prod_{k=1}^{j-1}\frac{(a-b)p-k}{bp+k} \equiv_{p^2} (a-b)\binom{a}{b}\frac{p(-1)^{j-1}}{j} \equiv_{p^2} (a-b)\binom{a}{b}\binom{p}{j},$$

and therefore

$$\binom{ap}{bp-j} = \binom{ap}{(a-b)p+j} \equiv_{p^2} b\binom{a}{b}\binom{p}{j}.$$

For (5.2), use Vandermonde-Chu's identity and (5.1) so that

$$\begin{aligned}\binom{p(n+m)+r}{pm+r} &= \sum_{j=0}^{r}\binom{p(n+m)}{pm+j}\binom{r}{r-j}\\ &\equiv_{p^2} \binom{n+m}{m}+n\binom{n+m}{m}\sum_{j=1}^{r}\binom{p}{j}\binom{r}{r-j}\\ &\equiv_{p^2} \binom{n+m}{m}\left(1+n\left(\binom{p+r}{r}-1\right)\right).\end{aligned}$$

In a similar way, we prove (5.3) as follows:

$$\begin{aligned}\binom{2pm+2r}{pm+r} &= \sum_{j=-r}^{r}\binom{2pm}{pm+j}\binom{2r}{r-j}\\ &= \binom{2pm}{pm}\binom{2r}{r} + \sum_{j=1}^{r}\binom{2pm}{pm+j}\binom{2r}{r-j} + \sum_{j=1}^{r}\binom{2pm}{pm-j}\binom{2r}{r+j}\\ &\equiv_{p^2} \binom{2m}{m}\left(\binom{2r}{r} + m\sum_{j=1}^{r}\binom{p}{j}\binom{2r}{r-j} + m\sum_{j=1}^{r}\binom{p}{p-j}\binom{2r}{r+j}\right)\\ &\equiv_{p^2} \binom{2m}{m}\left(\binom{2r}{r} + 2m\left(\binom{p+2r}{r} - \binom{2r}{r}\right)\right).\end{aligned}$$

Moreover,

$$\begin{aligned}\binom{3pm+3r}{pm+r} &= \sum_{j=-2r}^{r}\binom{3pm}{pm+j}\binom{3r}{r-j}\\ &= \binom{3pm}{pm}\binom{3r}{r} + \sum_{j=1}^{r}\binom{3pm}{pm+j}\binom{3r}{r-j} + \sum_{j=1}^{2r}\binom{3pm}{pm-j}\binom{3r}{r+j}\\ &\equiv_{p^2} \binom{3m}{m}\left(\binom{3r}{r} + 2m\left(\binom{p+3r}{r} - \binom{3r}{r}\right)\right) + \sum_{j=1}^{2r}\binom{3pm}{pm-j}\binom{3r}{r+j}.\end{aligned}$$

Now, (5.4) is equal to

$$\begin{aligned}&\sum_{j=1}^{p-1}\binom{3pm}{pm-j}\binom{3r}{r+j} + \binom{3pm}{pm-p}\binom{3r}{r+p} + \sum_{j=p+1}^{2r}\binom{3pm}{pm-j}\binom{3r}{r+j}\\ &= \sum_{j=1}^{p-1}\binom{3pm}{pm-j}\binom{3r}{r+j} + \binom{3pm}{pm-p}\binom{3r}{r+p} + \sum_{j=1}^{2r-p}\binom{3pm}{p(m-1)-j}\binom{3r}{r+p+j}\\ &\equiv_{p^2} m\binom{3m}{m}\left(\binom{p+3r}{p+r} - \binom{3r}{r} - \binom{3r}{r+p}\right) + \binom{3m}{m-1}\binom{3r}{r+p}\\ &\quad + (m-1)\binom{3m}{m-1}\sum_{j=1}^{2r-p}\binom{p}{p-j}\binom{3r}{r+p+j}\\ &\equiv_{p^2} m\binom{3m}{m}\left(\binom{p+3r}{2r} - \binom{3r}{r} - \binom{3r}{p+r}\right) + \binom{3m}{m-1}\binom{3r}{p+r}\\ &\quad + (m-1)\binom{3m}{m-1}\left(\binom{p+3r}{2p+r} - \binom{3r}{p+r}\right).\end{aligned}$$

The proof of the last congruence in (5.5) is analogous and hence is omitted here. □

Proof. We provide an alternative proof of Lemma 5.1 by reviving certain results found in [10] as equations (26) and (27), respectively. These are stated follows. If $n = n_1p + n_0$ and

$k = k_1p + k_0$ where $0 < n_0, k_0 < p$ then

$$\binom{np}{k} \equiv_{p^2} n\binom{n-1}{k_1}\binom{p}{k_0}, \tag{5.6}$$

$$\binom{n}{k} \equiv_{p^2} \binom{n_1}{k_1}\left[(1+n_1)\binom{n_0}{k_0} - (n_1+k_1)\binom{n_0-p}{k_0} - k_1\binom{n_0-p}{k_0+p}\right]. \tag{5.7}$$

For (5.1) of the lemma, apply (5.6) with $n_1 = a, n_0 = 0, k_1 = b, k_0 = j$. So,

$$\binom{ap}{bp+j} \equiv_{p^2} a\binom{a-1}{b}\binom{p}{j} = (a-b)\binom{a}{b}\binom{p}{j}.$$

For (5.2), apply (5.7) with $n_1 = n+m, n_0 = r = k_0, k_1 = m$. So,

$$\binom{p(n+m)+r}{pm+r} \equiv_{p^2} \binom{n+m}{m}\left[(1+m+n)\binom{r}{r} - (n+2m)\binom{r-p}{r} - m\binom{r-p}{r+p}\right]$$

To put this in the desired format consider applying (5.7) to $\binom{p+r}{r} \equiv_{p^2} 2 - \binom{r-p}{p}$ (with $n_1 = 1, n_0 = k_0 = r, k_1 = 0$); to $\binom{r-p}{r+p} = \binom{-p+r}{-2p} \equiv_{p^2} -3 + 2\binom{r-p}{r}$ (with $n_1 = -1, n_0 = r, k_1 = -2, k_0 = 0$). After substitution and simplifications, the desired outcome is reached.

For (5.3), apply (5.7) with $n_1 = 2m, n_0 = 2r, k_1 = m, k_0 = r$. So,

$$\binom{2pm+2r}{pm+r} \equiv_{p^2} \binom{2m}{m}\left[(1+2m)\binom{2r}{r} - 3m\binom{2r-p}{r} - m\binom{2r-p}{r+p}\right].$$

Let's reformulate this to get the result as stated in the lemma. To this end, employ (5.7) to $\binom{p+2r}{r} \equiv_{p^2} 2\binom{2r}{r} - \binom{2r-p}{r}$ (with $n_1 = 1, n_0 = 2r, k_1 = 0, k_0 = r$); to $\binom{p+2r}{r} = \binom{p+2r}{p+r} \equiv_{p^2} 2\binom{2r}{r} - 2\binom{2r-p}{r} - \binom{2r-p}{r+p}$ (with $n_1 = k_1 = 1, n_0 = 2r, k_0 = r$). Routine substitution completes the argument.

The congruence (5.4) demands a careful analysis. The setup begins by expressing $3r = \epsilon p + d$ where $0 < d < p$ and $\epsilon \in \{0, 1, 2\}$ which correspond to $0 < 3r < p, p < 3r < 2p$ and $2p < 3r < 3p$, respectively. Here, $\epsilon = \lfloor \frac{3r}{p} \rfloor$

Let $n_1 = 3m + \epsilon, n_0 = d, k_1 = m, k_0 = r$ and implement (5.7). So,

$$\binom{p(3m+\epsilon)+d}{pm+r} \equiv_{p^2} \binom{3m+\epsilon}{m}\left[(3m+\epsilon+1)\binom{d}{r} - (4m+\epsilon)\binom{d-p}{r} - m\binom{d-p}{r+p}\right].$$

Next, engage (5.6) with (with $n_1 = \epsilon, n_0 = d, k_1 = 0, k_0 = r$ to get

$$\binom{3r}{r} = \binom{\epsilon p + d}{r} \equiv_{p^2} (\epsilon+1)\binom{d}{r} - \epsilon\binom{d-p}{r};$$

with $n_1 = \epsilon + 1, n_0 = d, k_1 = 0, k_0 = r$ to get

$$\binom{p+3r}{r} = \binom{(\epsilon+1)p+d}{r} \equiv_{p^2} (\epsilon+2)\binom{d}{r} - (\epsilon+1)\binom{d-p}{r};$$

with $n_1 = \epsilon + 1, n_0 = d, k_1 = 1, k_0 = r$ to get

$$\binom{p+3r}{2r} = \binom{(\epsilon+1)p+d}{p+r} \equiv_{p^2} (\epsilon+1)(\epsilon+2)\binom{d}{r} - (\epsilon+1)(\epsilon+2)\binom{d-p}{r} - (\epsilon+1)\binom{d-p}{r+p}.$$

After proper substitutions, the result becomes

$$\begin{aligned}\binom{3pm+3r}{pm+r} &\equiv_{p^2} \binom{3m+\epsilon}{m}\binom{3r}{r} \\ &\qquad + \binom{3m+\epsilon}{m}\left(m\left[\frac{1}{\epsilon+1}\binom{p+3r}{2r}-\binom{3r}{r}\right]+2m\left[\binom{p+3r}{r}-\binom{3r}{r}\right]\right).\end{aligned}$$

For (5.5), apply (5.6) with $n_1 = n-1, n_0 = p-1, k_1 = 3m+\epsilon, k_0 = d-1$. Follow this through using $\binom{-1}{j} = (-1)^j$. The outcome is:

$$\begin{aligned}\binom{pn}{3pm+3r} &= \frac{pn}{3pm+3r}\binom{p(n-1)+p-1}{p(3m+\epsilon)+d-1} \\ &\equiv_{p^3} \frac{pn}{3pm+3r}\binom{n-1}{3m+\epsilon}\left[n\binom{p-1}{3r-1-\epsilon p}+(-1)^{r-\epsilon}(n-1)\right].\end{aligned} \tag{5.8}$$

Although doable, we opt to leave this congruence in its present form instead of committing to transform it into (5.5) because (5.8) will be more convenient for our subsequent calculations. □

Corollary 5.2. *For $p>3$ a prime and an integer $0 \le r < p$, we have the congruence*

$$\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r} \equiv_{p^2} \binom{3m}{m,m,m}\binom{3r}{r,r,r}\left[1+3pm(H_{3r}-H_r)\right].$$

Proof. This is a consequence of Lemma 5.1 and (5.7). However, we offer a more direct approach. Since $(pm+k)^{-1} \equiv_{p^2} \frac{1}{k}\left(1-\frac{pm}{k}\right)$, we obtain $(pm+k)^{-3} \equiv_{p^2} \frac{1}{k^3}\left(1-\frac{pm}{k}\right)^3 \equiv_{p^2} \frac{1}{k^3}\left(1-\frac{3pm}{k}\right) = \frac{1}{k^4}(k-3pm)$. For notational simplicity, denote $\binom{3j}{j,j,j} = \binom{3j}{j}\binom{2j}{j}$ by $\binom{3j}{j^3}$. We consider the expansion $\prod_{i=1}^n(\lambda_i+x) = \sum_{j=0}^n e_j(\lambda)x^{n-j}$ as our running theme, where e_j is the j-th *elementary symmetric function* in the parameters $\lambda = (\lambda_1,\dots,\lambda_n)$. In particular, $e_n = 1$ and $e_{n-1}(1,\dots,n) = n!H_n$. The claim then follows from

$$\begin{aligned}\binom{3pm+3r}{(pm+r)^3} &= \binom{3pm}{(pm)^3}\prod_{j=1}^{3r}(j+3pm)\prod_{k=1}^{r}(pm+k)^{-3} \\ &\equiv_{p^2} \binom{3pm}{(pm)^3}\frac{1}{r!^4}\prod_{j=1}^{3r}(j+3pm)\prod_{k=1}^{r}(k-3pm) \\ &\equiv_{p^2} \binom{3pm}{(pm)^3}\frac{1}{r!^4}(3r)!r!\left[1+3pmH_{3r}-3pmH_r\right].\end{aligned}$$

□

This fact is even more general as stated below but its proof is left to the interested reader.

Exercise 5.3. If $A>0, 0 \le r < p$ are integers and $p>3$ a prime, then

$$\binom{Apm+Ar}{pm+r,\dots,pm+r} := \frac{(Apm+Ar)!}{(pm+r)!^A} \equiv_{p^2} \binom{Am}{m,\cdots,m}\binom{Ar}{r,\cdots,r}\left[1+Apm(H_{Ar}-H_r)\right].$$

Corollary 5.4. *For $p>3$ a prime and an integer $0 \le r < p$, we have*

$$\binom{p(n+m)+r}{pm+r} \equiv_{p^2} \binom{n+m}{m}[1+pnH_r].$$

Proof. It is easy to check that $\binom{p+r}{r} = \frac{1}{r!}\prod_{j=1}^{r}(p+j) \equiv_{p^2} 1 + pH_r$. The rest follows from (5.2) of Lemma 5.1. □

Corollary 5.5. *Let $N = n - 3m$. For $p > 3$ a prime and an integer $0 < r < p$, it holds that*

$$\binom{pn}{3pm+3r} \equiv_{p^3} \left(\frac{p}{3r} - \frac{p^2 m}{3r^2}\right)(-1)^r \binom{n}{3m} \cdot \begin{cases} N(-1 + pnH_{3r-1}), & \text{if } 0 < r < \frac{p}{3} \\ \binom{N}{2}\dfrac{2(1 - pnH_{3r-1-p})}{3m+1}, & \text{if } \frac{p}{3} < r < \frac{2p}{3} \\ \binom{N}{3}\dfrac{6(-1 + pnH_{3r-1-2p})}{(3m+1)(3m+2)}, & \text{if } \frac{2p}{3} < r < p. \end{cases}$$

Proof. We continue where we left off (5.8) with $\epsilon = \lfloor \frac{3r}{p} \rfloor$. That is,

$$\binom{pn}{3pm+3r} \equiv_{p^3} \frac{pn}{3pm+3r}\binom{n-1}{3m+\epsilon}\left[n\binom{p-1}{3r-1-\epsilon p} + (-1)^{r-\epsilon}(n-1)\right].$$

Combining this step and the easy facts $\frac{1}{3pm+3r} \equiv_{p^2} \frac{1}{3r} - \frac{pm}{3r^2}$, $\binom{p-1}{j} \equiv_{p^2} (-1)^j[1 - pH_j]$, we reach the desired conclusion. □

Lemma 5.6. *If $p > 3$ is a prime then*

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r} \equiv_{p^2} -3q_p(1/3) + \frac{3p}{2}q_p(1/3)^2, \tag{5.9}$$

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2} \equiv_p -\frac{9}{2}q_p(1/3)^2, \tag{5.10}$$

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{(H_{3r} - H_r)3^{-3r}}{r} \equiv_p 0. \tag{5.11}$$

Proof. By (2.1), $q_p(1/27) \equiv_{p^2} 3\,q_p(1/3) + 3p\,q_p(1/3)^2$. Therefore, by (5) in [11, Theorem 4],

$$\begin{aligned}\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r} = \sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r} &\equiv_{p^2} -q_p(1/27) + \frac{p}{2}q_p(1/27)^2 \\ &\equiv_{p^2} -3q_p(1/3) + \frac{3p}{2}q_p(1/3)^2.\end{aligned}$$

In a similar way, by (6) in [11, Theorem 4],

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2} = \sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r^2} \equiv_p -\frac{1}{2}q_p(1/27)^2 \equiv_p -\frac{9}{2}q_p(1/3)^2.$$

By (1) in [11, Theorem 1],

$$\frac{(1/3)_r(2/3)_r}{(1)_r^2}\sum_{j=0}^{r-1}\left(\frac{1}{1/3+j} + \frac{1}{2/3+j}\right) = \sum_{k=0}^{r-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{r-k}.$$

Hence (5.11) is implied by the following

$$\begin{aligned}\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{(3H_{3r}-H_r)3^{-3r}}{r} &= \sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r}\cdot\sum_{j=0}^{r-1}\left(\frac{1}{1/3+j}+\frac{1}{2/3+j}\right)\\
&= \sum_{r=1}^{p-1}\frac{1}{r}\sum_{k=0}^{r-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{r-k}\\
&= \sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\sum_{r=k+1}^{p-1}\frac{1}{r(r-k)}\\
&= \sum_{r=1}^{p-1}\frac{1}{r^2}+\sum_{k=1}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\left(\frac{1}{k}\sum_{r=k+1}^{p-1}\left(\frac{1}{r-k}-\frac{1}{r}\right)\right)\\
&\equiv_p \sum_{k=1}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{k}\left(H_{p-1-k}-H_{p-1}+H_k\right)\\
&\equiv_p \sum_{k=1}^{p-1}\binom{3k}{k,k,k}\frac{2H_k\,3^{-3k}}{k},\end{aligned}$$

because $H_{p-1-k}\equiv_p H_k$ and $H_{p-1}\equiv_p \sum_{r=1}^{p-1}\frac{1}{r^2}\equiv_p\sum_{j=1}^{p-1}j\equiv_p 0$ as $p\neq 2$. □

6. Proof of Conjecture 1.1

In this section, we combine the results from the preceding sections to arrive at a proof for Theorem 4.1 (restated here for the reader's convenience) and therefore for Conjecture 1.1.

Theorem 6.1. *For a prime $p>3$ and $m,n\in\mathbb{N}^+$, we have*

$$\begin{aligned}&\sum_{r=1}^{p-1}(-1)^r\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r}\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}3^{-3r}\\
&\qquad\equiv_{p^3} p\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}q_p(3^{-(n-3m)}).\end{aligned}$$

Proof. Based on Corollaries 5.2, 5.4, 5.5 and the congruence (2.1), the assertion is equivalent to

$$\begin{aligned}&\sum_{r=1}^{p-1}\binom{3r}{r,r,r}(1+3pm(H_{3r}-H_r))(1+pnH_r)\left(\frac{1}{3r}-\frac{pm}{3r^2}\right)B_r(p,n,m)3^{-3r}\\
&\qquad\equiv_{p^2} q_p(1/3)+\frac{p(N-1)}{2}q_p(1/3)^2;\end{aligned}\tag{6.1}$$

where

$$B_r(p,n,m)=\begin{cases}-1+pnH_{3r-1}, & \text{if } 0<r<\frac{p}{3}\\[4pt] \dfrac{(N-1)(1-pnH_{3r-1-p})}{3m+1}, & \text{if } \frac{p}{3}<r<\frac{2p}{3}\\[4pt] \dfrac{(N-1)(N-2)(-1+pnH_{3r-1-2p})}{(3m+1)(3m+2)}, & \text{if } \frac{2p}{3}<r<p.\end{cases}$$

Now we split the sum on the left-hand side of (6.1) into three pieces according as

$$S_1=\sum_{r=1}^{\lfloor p/3\rfloor}(\cdot),\qquad S_2=\sum_{r=\lceil p/3\rceil}^{\lfloor 2p/3\rfloor}(\cdot),\qquad\text{and}\qquad S_3=\sum_{r=\lceil 2p/3\rceil}^{p-1}(\cdot).$$

As regards S_1,

$$S_1\equiv_{p^2}\frac{1}{3}\sum_{r=1}^{\lfloor p/3\rfloor}\binom{3r}{r,r,r}\left(-\frac{1}{r}-\frac{pN}{3r^2}+\frac{pN(H_{3r}-H_r)}{r}\right)3^{-3r}.$$

If $\frac{p}{3}<r<\frac{2p}{3}$ then $\binom{3r}{r,r,r}\equiv_p 0$ and $1+3pm(H_{3r}-H_r)\equiv_p 1+3m$ with $B_r(p,n,m)\equiv_p\frac{(N-1)}{(3m+1)}$. These imply that

$$\begin{aligned}S_2&\equiv_{p^2}\sum_{r=\lceil p/3\rceil}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}(1+3pm(H_{3r}-H_r))(1+pnH_r)\left(\frac{1}{3r}-\frac{pm}{3r^2}\right)B_r(p,n,m)3^{-3r}\\&\equiv_{p^2}\sum_{r=\lceil p/3\rceil}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}(1+3m)\left(\frac{1}{3r}\right)\frac{(N-1)}{(3m+1)}3^{-3r}\\&\equiv_{p^2}\frac{(N-1)}{3}\sum_{r=\lceil p/3\rceil}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}.\end{aligned}$$

Finally, we have that $S_3\equiv_{p^2}0$ because obviously $\binom{3r}{r,r,r}\equiv_{p^2}0$ as long as $\frac{2p}{3}<r<p$.

Again $\binom{3r}{r,r,r}\equiv_p 0$ if $\frac{p}{3}<r<\frac{2p}{3}$ and $\binom{3r}{r,r,r}\equiv_{p^2}0$ if $\frac{2p}{3}<r<p$. So, from Lemma 5.6 we know

$$\begin{aligned}\sum_{r=1}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}&\equiv_{p^2}\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}\equiv_{p^2}-3q_p(1/3)+\frac{3p}{2}q_p(1/3)^2,\\ p\sum_{r=1}^{\lfloor p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2}&\equiv_{p^2}p\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2}\equiv_{p^2}-\frac{9p}{2}q_p(1/3)^2.\end{aligned}$$

As before $\binom{3r}{r,r,r}\equiv_{p^2}0$ for $\frac{2p}{3}<r<p$. As well as $\binom{3r}{r,r,r}\equiv_p 0$ and $pH_{3r}-pH_r\equiv_p 1$ for $\frac{p}{3}<r<\frac{2p}{3}$. Therefore, by Lemma 5.6

$$\begin{aligned}0\equiv_{p^2}p\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{(H_{3r}-H_r)3^{-3r}}{r}&\equiv_{p^2}p\sum_{r=1}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{(H_{3r}-H_r)3^{-3r}}{r}\\&\equiv_{p^2}p\sum_{r=1}^{\lfloor p/3\rfloor}\binom{3r}{r,r,r}\frac{(H_{3r}-H_r)3^{-3r}}{r}+\sum_{r=\lceil p/3\rceil}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}.\end{aligned}$$

Putting all these together, we conclude that

$$\begin{aligned}
S_1+S_2+S_3 &\equiv_{p^2} \frac{1}{3}\sum_{r=1}^{\lfloor p/3\rfloor}\binom{3r}{r,r,r}\left(-\frac{1}{r}-\frac{pN}{3r^2}+\frac{pN(H_{3r}-H_r)}{r}\right)3^{-3r}\\
&\quad+\frac{(N-1)}{3}\sum_{r=\lceil p/3\rceil}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}+0\\
&\equiv_{p^2} -\frac{1}{3}\sum_{r=1}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}-\frac{N}{9}\sum_{r=1}^{\lfloor p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2}+\frac{N}{3}\cdot 0\\
&\equiv_{p^2} -\frac{1}{3}\left(-3q_p(1/3)+\frac{3p}{2}q_p(1/3)^2\right)-\frac{N}{9}\left(-\frac{9p}{2}q_p(1/3)^2\right)\\
&\equiv_{p^2} q_p(1/3)+\frac{p(N-1)}{2}q_p(1/3)^2,
\end{aligned}$$

which is exactly what we expect. The proof is complete. □

7. Conclusions and Remarks

In this final section, we extend the congruence on $a_i(n)$ (for $i>0$), discussed in the earlier sections, from modulo p^2 to modulo p^3. While stating our claim in its generality, we only exhibit proof outlines for the case $i=1$ as a prototypical example. We believe the curious researcher would be able to account for the remaining cases.

Conjecture 7.1. *For $n,i\in\mathbb{N}^+$ and a prime $p>2i$,*

$$a_i(pn)\equiv_{p^3}(-1)^{i-1}\frac{a_1(pn)}{i^2\binom{2i-1}{i-1}}\equiv_{p^3}\frac{(-1)^{i-1}p^2\binom{n+2}{2}a_1(n)}{i^2\binom{2i-1}{i-1}}.$$

Proof. Ingredients for $a_1(pn)\equiv_{p^3} p^2\binom{n+2}{2}a_1(n)$.

(A) By partial fraction decomposition

$$\begin{aligned}
a_i(n) &= \frac{1}{3^i}\sum_{k=0}^{n-1}(-1)^{n-k}\binom{3k}{k}\binom{2k}{k}\binom{n}{3k}\binom{n+k}{k}\frac{\binom{n-3k}{i}3^{n-3k}}{\binom{k+i}{i}}\\
&= (-1)^i a_0(n)+\frac{i}{3^i}\sum_{j=1}^{i}(-1)^{j-1}\binom{i-1}{j-1}\binom{n+3j}{i}b_j(n)
\end{aligned}$$

where for $j\in\mathbb{N}^+$,

$$b_j(n):=\sum_{k=0}^{n-1}(-1)^{n-k}(n-3k)\binom{3k}{k}\binom{2k}{k}\binom{n}{3k}\binom{n+k}{k}\frac{3^{n-3k}}{k+j}.$$

Thus, $a_0(np)\equiv_{p^3} a_0(n)$ implies

$$\begin{aligned}
a_1(np) &= -a_0(np)+\frac{np+3}{3}b_1(np)\equiv_{p^3} -a_0(n)+\frac{np+3}{3}b_1(np)\\
&\equiv_{p^3} a_1(n)+\frac{np+3}{3}b_1(np)-\frac{p+3}{3}b_1(p).
\end{aligned}$$

(B) Hence, it suffices to show that

$$b_1(np) \equiv_{p^3} \frac{3}{np+3}\left(p^2\binom{n+2}{2}-1\right)a_1(n)+\frac{n+3}{np+3}\,b_1(n),$$

or, since $a_1(n) = -a_0(n) + (n+3)b_1(n)/3$,

$$b_1(np) \equiv_{p^3} p^2\binom{n+3}{3}b_1(n)+\left(1-\frac{pn}{3}-\frac{p^2(n+3)(7n+6)}{18}\right)a_0(n). \tag{7.1}$$

(C) The above congruence is implied by the following

$$\begin{aligned}\sum_{r=0}^{p-1}(-1)^r\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r}\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}\frac{3^{-3r}}{pm+r+1}\\ \equiv_{p^3}\left(\frac{p^2}{m+1}\binom{n+3}{3}+1-\frac{pn}{3}-\frac{p^2(n+3)(7n+6)}{18}\right)\\ \cdot\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}3^{-N(p-1)}\end{aligned} \tag{7.2}$$

By summing over m, it is immediate to recover (7.1).

(D) In order to prove (7.2), we have the old machinery, $\frac{1}{pm+r+1}\equiv_{p^2}\frac{1}{r+1}-\frac{mp}{(r+1)^2}$, and

$$\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r+1}=\frac{9p}{2}\binom{3p}{p,p,p}3^{-3p}\equiv_{p^2}\; p-3p^2q_p(1/3),$$

$$\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{(r+1)^2}=\frac{9(9p+2)}{4}\binom{3p}{p,p,p}3^{-3p}-\frac{9}{2}\equiv_p-\frac{7}{2}.$$

(E) Finally, we can modify a previous proof as follows:

$$\begin{aligned}\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{(3H_{3r}-H_r)3^{-3r}}{r+1}&=\sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r+1}\cdot\sum_{j=0}^{r-1}\left(\frac{1}{1/3+j}+\frac{1}{2/3+j}\right)\\ &=\sum_{r=1}^{p-1}\frac{1}{r+1}\sum_{k=0}^{r-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{r-k}\\ &=\sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\sum_{r=k+1}^{p-1}\frac{1}{(r+1)(r-k)}\\ &=\sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\left(\frac{1}{k+1}\sum_{r=k+1}^{p-1}\left(\frac{1}{r-k}-\frac{1}{r+1}\right)\right)\\ &=\sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{H_{p-1-k}-H_p+H_{k+1}}{k+1}\\ &\equiv_p\sum_{k=0}^{p-1}\binom{3k}{k,k,k}\frac{(H_k-H_p+H_{k+1})\,3^{-3k}}{k+1},\end{aligned}$$

which implies that

$$\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{(H_{3r}-H_r)3^{-3r}}{r+1} \equiv_p \frac{1}{3}\sum_{k=0}^{p-1}\binom{3k}{k,k,k}\frac{(-1/p+1/(k+1))\,3^{-3k}}{k+1}$$
$$\equiv_p \frac{1}{3}\left(-1-\frac{7}{2}\right) = -\frac{3}{2}.$$

□

Remark 7.2. We showed that the conjecture $a_0(pn) \equiv_{p^3} a_0(n)$ holds true. Although it is not pursued here, the techniques established in this paper if combined with existing literature on supercongruences (see references below) for binomials of the type $\binom{p^r n+k}{p^t m+j}$, there is enough reliable verity to believe that $a_0(p^r n) \equiv_{p^{3r}} a_0(p^{r-1}n)$ should be within easy grasp.

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